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A SIGNIFICANCE TEST FOR MINIMUM RANK IN FACTOR ANALYSIS

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This paper presents a new method of determining the minimum rank in factor analysis, appropriate to the principal axes solution. The new method is compared with a former method which, with some adjustment, is more convenient for the centroid approach. Both methods are applied to two familiar examples.

The question of when to stop in the process of extracting factors from a correlation matrix seems to be answered in common practice by calculating the mean absolute residual correlation and comparing it with the standard error of a simple correlation coefficient. Unfortunately, this test is not used correctly under the assumptions implied by its use; and even if it were, it would not ordinarily give a satisfactory answer. There is a tendency for psychologists using this test to underestimate the minimum number of significant factors needed to describe their data. This is particularly true if the correlations are based on at most 200 observations, because under such circumstances the standard error of a correlation coefficient is large, while it takes comparatively few factors to reduce a set of positive correlations to a set of residuals satisfying the customary standard error criterion. As an illustration of the situation, a 10-rowed matrix containing positive elements drawn from a table of random numbers was factored by the centroid method. The values of the mean absolute residuals turned out to be .477, .213, .176, .135, .122, .075, .067, corresponding to the extraction of 0, 1, 2, 3, 4, 5, 6, factors respectively. If common practice were followed, not over 4 factors would be considered significant for a sample of 100, and not over 5 for a sample of 200.

In a previous paper (1) the author presented a method for handling this question, having particularly in mind factor analysis based upon the principal axes approach. However, with a slight change, this method turns out to be more suitable for the centroid approach. The present paper develops another method which is more convenient when principal axes are used, and which is interesting for its own sake

because of the relationship it brings out between these two approaches to the subject. The paper proceeds by developing this second method, making the necessary change in the first method, and then applying both methods to two well-known examples.

SECOND METHOD

Consider a set of n statistical variables $z_1, z_2, \cdots, z_n$, with zero means, which are subject to a significant error of measurement. Each variable z_i may be thought of as the sum of a true variable x_i and an error variable y_i. It will be assumed that the error variables are uncorrelated with each other and with the true variables. With these assumptions, and with E denoting the expected value, it follows readily that

$$m_{ij} = E(z_i z_j) = E(x_i + y_i)(x_j + y_j) = E(x_i x_j) = m'_{ij} \quad (i \neq j),$$

$$m_{ii} = E(z_i^2) = E(x_i^2 + 2x_i y_i + y_i^2) = E(x_i^2) + E(y_i^2)$$

$$= m'_{ii} + m''_{ii},$$

$$m'_{ii} = m_{ii} - m''_{ii} = m_{ii}\left(1 - \frac{m''_{ii}}{m_{ii}}\right) = m_{ii}(1 - \beta_i),$$

$$r'_{ij} = \frac{m'_{ij}}{\sqrt{m'_{ii} m'_{jj}}} = \frac{m_{ij}}{\sqrt{m_{ii}(1-\beta_i)\, m_{jj}(1-\beta_j)}}$$

$$= \frac{r_{ij}}{\sqrt{(1-\beta_i)(1-\beta_j)}} \qquad (i \neq j). \qquad (1)$$

The notation here is somewhat simpler than that customarily employed with the centroid method. Here z_i is the test score, x_i the common factor part of it, and y_i the specific and error factor parts combined. Hence r'_{ij} is the correlation between the common factor parts of tests i and j, while r_{ij} is the usual correlation. In the customary centroid notation, $r'_{ij} = r_{ij}/h_i h_j$, where h_i^2 is the communality for test i; consequently $h_i^2 = 1 - \beta_i$ expresses the relationship between the present notation and the usual centroid notation.

The test obtained in this paper hinges upon an inequality found in studying properties of the common factor correlation determinant $|r'_{ij}|$. By means of (1), $|r'_{ij}|$ may be expressed in terms of the r_{ij} and β_i. If the denominators are factored out from each row and column of $|r'_{ij}|$, it can be expressed as $1/(1-\beta_1)(1-\beta_2)\cdots(1-\beta_n)$ multiplied by the determinant

$$F = \begin{vmatrix} 1-\beta_1 & r_{12} & \cdots & r_{1n} \\ r_{21} & 1-\beta_2 & \cdots & r_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ r_{n1} & r_{n2} & \cdots & 1-\beta_n \end{vmatrix} \tag{2}$$

Considered as a function of the β_i, F may be expanded by means of Taylor's formula about some convenient point $P(\bar{\beta}_1, \bar{\beta}_2, \cdots, \bar{\beta}_n)$. The derivatives of F are found easily since F is linear in any one variable. Thus $\frac{\partial F}{\partial \beta_i} = F_{)i(}$, where $F_{)i(}$ denotes the minor of F obtained by deleting row and column i, $\frac{\partial^2 F}{\partial \beta_i \partial \beta_j} = F_{)ij(}$ $(i \neq j)$, where $F_{)ij(}$ is the minor obtained by deleting rows and columns i and j. Substituting these and corresponding formulas in Taylor's expansion for n variables

$$F = \bar{F} - \sum_1^n \bar{F}_{)i(}(\beta_i - \bar{\beta}_i) + \sum_{i<j} \bar{F}_{)ij(}(\beta_i - \bar{\beta}_i)(\beta_j - \bar{\beta}_j) + \cdots + (-1)^n(\beta_1 - \bar{\beta}_1)\cdots(\beta_n - \bar{\beta}_n), \tag{3}$$

where $\bar{F}$ is the value of F at P, etc.

The equation $F = 0$ represents a surface in a space of n dimensions. In order to study this surface, consider its intersections with a line L passing through P and having direction numbers $a_1, a_2, \cdots, a_n$, all of which are positive. The equation of L may be written as

$$\beta_i - \bar{\beta}_i = ta_i \qquad (i = 1, 2, \cdots, n). \tag{4}$$

To find the points of intersection of L with the surface, it is necessary to substitute (4) in (3) and solve the resulting equation (5) for t. Thus

$$0 = \bar{F} - t\sum \bar{F}_{)i(}a_i + t^2 \sum_{i<j} \bar{F}_{)ij(}a_i a_j + \cdots + (-t)^n a_1 \cdots a_n . \tag{5}$$

Now the nature of the surface $F = 0$ will be determined by the nature of the roots of (5).

First, it will be shown that the roots are real. Consider the

symmetric determinant

$$G = \begin{vmatrix} \dfrac{1-\bar{\beta}_1}{\sqrt{a_1 a_1}} & \cdots\cdots & \dfrac{r_{1n}}{\sqrt{a_1 a_n}} \\ \vdots & & \vdots \\ \dfrac{r_{n1}}{\sqrt{a_1 a_n}} & \cdots\cdots & \dfrac{1-\bar{\beta}_n}{\sqrt{a_n a_n}} \end{vmatrix}$$

If t is subtracted from each diagonal element of G, and the resulting determinant, H, is expanded by means of (3) about the origin, where now $\beta_1 = \beta_2 = \cdots = \beta_n = t$,

$$H = G - t \sum G_{)i(} + t^2 \sum_{i<j} G_{)ij(} + \cdots + (-t)^n .$$

But upon factoring out the denominators in G, $G_{)i(}$, etc., it becomes evident that

$$G = \frac{\bar{F}}{a_1 a_2 \cdots a_n} , \qquad G_{)i(} = \frac{\bar{F}_{)i(}\, a_i}{a_1 a_2 \cdots a_n} , \cdots .$$

Hence,

$$H = \frac{1}{a_1 a_2 \cdots a_n} [\bar{F} - t \sum \bar{F}_{)i(}\, a_i + t^2 \sum_{i<j} \bar{F}_{)ij(}\, a_i\, a_j + \cdots + (-t)^n a_1 \cdots a_n].$$

This shows that the roots of (5) are identical with those of $H = 0$, which is the characteristic equation of G. Since it is well known that the roots of the characteristic equation of a real symmetric matrix are all real, the roots of (5) must be real for every point P and every set of positive numbers a_i.

Secondly, it is known that the n roots of (5) are continuous functions of the $\bar{\beta}_i$ and the a_i which enter in the coefficients of the equation and which serve to determine the line L.

Because of these two properties, it follows that every line with positive direction numbers intersects the surface in exactly n real points, if a root of multiplicity q is considered as corresponding to q coincident points on the surface, and that the surface consists of n continuous sheets. If the a_i are held fixed and the $\bar{\beta}_i$ allowed to vary, the resulting system of parallel lines, by their intersections with the surface, will serve to describe these n sheets. These n sheets may not be distinct at all points, since (5) may not possess distinct roots for all values of the $\bar{\beta}_i$. However, if each sheet is numbered according to the

order of magnitude of the corresponding root of (5) where the roots are distinct, there is no ambiguity in speaking of the sheets as though they were distinct. Furthermore, the continuity of the roots insures that no line with positive direction numbers can intersect the same numbered sheet more than once. Otherwise there would exist a line parallel to the line in question intersecting the surface in, usually, two coincident points, in the neighborhood of which the roots of (5) would not all vary continuously. This excludes the possibility of any sheet being closed, say, like an ellipsoid. As a matter of fact, the continuity of the roots restricts the nature of the surface to the extent that no line with positive direction numbers can be tangent to it. These properties of the surface suggest the *Theorem*: If F is of rank $n - q$ for some point $P(\bar{\beta}_1, \cdots \bar{\beta}_n)$ in the all-positive region, the q-th largest characteristic root of the correlation matrix can not exceed the largest of the numbers $\bar{\beta}_i$.

Since F is of rank $n - q$ at P, all principal minors of more than $n - q$ rows vanish at this point. Hence all F's with less than q subscripts vanish and $t = 0$ will be a root of multiplicity q of (5). Consequently, at this point q of the n sheets of the surface coincide. It is sufficient to assume that this q-tuple root is the smallest root of the equation, otherwise the theorem could be proved true for a larger characteristic root. If the theorem were false, it would be possible to find a line with positive direction numbers which would pass through the point P at which q of the sheets coincide and the point on the q-th sheet where the line $\beta_1 = \beta_2 = \cdots = \beta_n$ intersects it to produce the q-th characteristic root. However, previous considerations exclude the possibility of any such line intersecting the same sheet twice.

If k_q represents the q-th characteristic root and $\bar{\beta}_m$ the largest of the numbers $\bar{\beta}_1, \cdots, \bar{\beta}_n$, then the theorem states that $k_q \leq \bar{\beta}_m$. Since $h_i^2 = 1 - \beta_i$, this is equivalent to

$$k_q + h_m^2 \leq 1, \tag{6}$$

where h_m^2 is the smallest of the communalities. This inequality is true for population parameters only. In order to apply it to a sample, it is necessary to consider its sampling error.

In general if $z = x + y$, then $\sigma_z^2 = \sigma_x^2 + 2r\sigma_x\sigma_y + \sigma_y^2$. If r is unknown but positive, the best that can be found as an upper bound for σ_z is $\sigma_z \leq \sigma_x + \sigma_y$. If $z \leq 1$, then for z approximately normally distributed, the odds are better than 19 to 1 that a sample value of z will not exceed $1 + 1.65(\sigma_x + \sigma_y)$.

Geometrical considerations show that sample values of h_m^2 and k_q are positively correlated, so that the above statements apply here.

Consequently, with temporary normality assumptions and with capital letters representing sample values, the odds are at least 19 to 1 that

$$K_q + H_m^2 \leq 1 + 1.65(\sigma_{K_q} + \sigma_{H_m^2}).$$

Now H_m^2 and K_q are both variances. If the basic variables, that is, the ability scores which produce the test scores, are assumed to be normally distributed, the standard error of a variance may be employed here to produce odds of at least 19 to 1 that

$$K_q + H_m^2 \leq 1 + 1.65\sqrt{2/N}\,(k_q + h_m^2). \tag{7}$$

For large samples the population values k_q and h_m^2 may be replaced by their sample values. With this substitution, the inequality finally takes the form

$$K_q + H_m^2 \leq \frac{1}{1 - 1.65\sqrt{2/N}}. \tag{8}$$

This inequality underestimates the actual relationship at several stages; consequently, sample values close to this significance level should be judged significant, while values beyond this level should be considered highly significant.

Comparison of Methods

The former method was based on Hotelling's paper (2) in which no specific factors were introduced, and hence must be altered slightly to fit the more general approach to the subject. In the notation of the author's first paper, $r'_{ij} = r_{ij}\sqrt{(1+\lambda_i)(1+\lambda_j)}$, where the λ_i were chosen for their mathematical convenience rather than their psychological meaning. In the present notation $r'_{ij} = \dfrac{r_{ij}}{h_i h_j}$. Hence, when specific factors are considered, $\lambda_i = \dfrac{1}{h_i^2} - 1$, as in contrast to the statement made in that paper that $\lambda_i = \dfrac{1}{r_i} - 1$, which is valid only when specific factors are ignored. Consequently, by merely replacing reliability coefficients (r_i) by communalities (h_i^2), the material of that paper becomes valid for the usual factor analysis setup. With this alteration, the results of that paper can be condensed into

$$W = \sqrt{N/2n}\,\left(\frac{Z}{\phi} - 1\right), \tag{9}$$

$$\frac{Z}{\phi} \geq \frac{|r_{ij}|}{\sum_{a_1<\ldots<a_n}(1-h^2_{a_1})\cdots(1-h^2_{a_q})h^2_{a_{q+1}}\cdots h^2_{a_n} + \sum_{a_1<\ldots<a_n}(1-h^2_{a_1})\cdots(1-h^2_{a_{q+1}})h^2_{a_{q+2}}\cdots h^2_{a_n}) + \cdots},$$

where the sums in the denominator continue until the final term $(1 - h_1^2) \cdots (1 - h_n^2)$ is reached, $|r_{ij}|$ is the correlation determinant with diagonal elements unity, N is the size sample, n is the number of tests, $n - q$ is the postulated minimum essential number of factors, and $W \geq 1.65$ produces odds of at least 19 to 1 against the truth of the postulate.

The results of the present paper can be condensed into

$$K_q + H_m^2 \leq \frac{1}{1 - 1.65\sqrt{2/N}},$$

where K_q is the q-th characteristic root of the correlation matrix with diagonal elements unity, H_m^2 is the smallest of the communalities, $n - q$ is the postulated minimum essential number of factors, and the odds are at least 19 to 1 against the truth of the postulate if the inequality is not satisfied.

The assumptions upon which the proofs of these two methods depend are quite similar. Each requires large samples and approximate normal distributions in the basic variables. The restrictions of the present method are perhaps less severe, but it requires larger samples to give one the same confidence in its conclusions. In each case it is necessary to replace population parameters by sample estimates.

Illustrative Examples

A. The following example was judged by Holzinger (3) to require but one significant factor.

$$\begin{vmatrix} 1 & .059 & .195 & .200 & .297 \\ & 1 & .149 & .284 & .249 \\ & & 1 & .466 & .669 \\ & & & 1 & .692 \\ & & & & 1 \end{vmatrix}$$

Calculations by standard methods give $|r_{ij}| = .239$; $H_i^2 = .243, .245, .675, .746, .797$; $K_i = 2.45, .95, .84, .53, .23$. Here $N = 140$ and $n = 5$.

Method 1. Assume one factor is sufficient. Then the rank is 1 and $q = 4$. Here the denominator of (9) is

$$\sum_{a_1<\ldots<a_5} (1 - H_{a_1}^2) \cdots (1 - H_{a_4}^2) H_{a_5}^2 + (1 - H_1^2) \cdots (1 - H_5^2) = .101,$$

and $W = \sqrt{140/2(5)}\ (2.36 - 1) = 5.1$.

This value of W is highly significant, so that there is no doubt but that the hypothesis is false, and hence that at least two factors are

necessary. If it is assumed that two factors are sufficient, a non-significant value of W is obtained.

Method 2. Assume one factor is sufficient. Then $K_4 = .95$, $H_m^2 = .24$, and (8) becomes

$$.95 + .24 \leq \frac{1}{1 - 1.65\sqrt{2/140}} = 1.25 .$$

This is barely satisfied. Because of the many inequalities used to obtain this final inequality, such a borderline case should be considered as significant. Although the conclusion that at least two factors are necessary is obtained by both methods, the first method arrives at the result decisively.

B. The following example was judged by both Holzinger and Thurstone to require but one significant factor.

$$\begin{vmatrix} 1 & .58 & .58 & .51 & .48 & .43 & .43 & .46 \\ & 1 & .47 & .50 & .53 & .40 & .44 & .45 \\ & & 1 & .51 & .48 & .41 & .45 & .38 \\ & & & 1 & .34 & .50 & .35 & .40 \\ & & & & 1 & .33 & .41 & .39 \\ & & & & & 1 & .41 & .33 \\ & & & & & & 1 & .31 \\ & & & & & & & 1 \end{vmatrix}$$

Calculations give $|r_{ij}| = .06$; $H_i^2 = .656, .639, .571, .587, .559, .522, .434, .377$; $K_i = 4.08, .77, \cdots$. Here $N = 127$ and $n = 8$.

Method 1. Assume one factor is sufficient. Then $q = 7$ and the denominator of (9) is

$$\sum_{a_1<\ldots<a_8} (1 - H_{a_1}^2) \cdots (1 - H_{a_7}^2) H_{a_8}^2 + (1 - H_1^2) \cdots (1 - H_8^2) = .18 .$$

The corresponding inequality becomes $Z/\phi \geq .33$. Since a value of $Z/\phi \leq 1$ cannot yield a significant result with this test, the hypothesis is not discredited here.

Method 2. Assume one factor is sufficient. Then $K_7 = .77$, $H_m^2 = .38$, and (8) becomes

$$.77 + .38 \leq \frac{1}{1 - 1.65\sqrt{2/127}} = 1.26 .$$

This is not as close as in the previous example, yet it is sufficiently close to make one doubt the truth of the hypothesis. Hence to be on the safe side, two factors should be considered.

CONCLUSIONS

The former method is convenient when the centroid technique is used, since the communalities are then available and it is only necessary to calculate the value of the correlation determinant. The present method is convenient when the principal axes technique is used, since the characteristic roots are then available and it is only necessary to find a suitable estimate of the smallest communality. The usual procedure of selecting the largest element in a given row as a first estimate of the communality for that row is not permissible here since such an estimate will invariably overestimate the true value. However, one of the more refined methods available will usually give a satisfactory estimate. A comparison of the two examples above indicates that the superiority of one test over the other varies with the problem. There are reasons, however, for believing that the first method is apt to be the more sensitive; and certainly it is easier to calculate the value of $|r_{ij}|$ than it is to calculate a number of characteristic roots if the technique employed does not yield them.

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NEURON CIRCUITS: THE SELF-EXCITING NEURON

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Conditions are determined under which a single self-exciting neuron can reach a state of permanent excitation, in the case that the neuron develops both an excitatory and an inhibitory substance as well as in the case that it develops only a single, excitatory substance. In case both substances are developed it is possible to have periodic solutions, a possibility which does not arise when the circuit consists of neurons, or of a neuron, developing only an excitatory substance.

1. Formulation of the problem. It is known that the nervous system contains numerous instances of pairs of mutually exciting neurons. Rashevsky (1, Ch. XXV) has shown theoretically that such a pair may be brought to a state of permanent excitation and so may furnish the necessary hysteresial element in the formation of a conditioned reflex. However, one finds instances also of single neurons possessing dendrites which reach forward and appear to make synaptic contacts with collaterals to their own axons, thus forming mononeuronic circuits.* It is the purpose of the present paper to discuss theoretically a circuit of this simple type (Fig. 1).

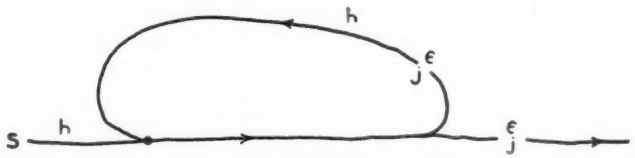

FIGURE 1

In notation and method we shall follow Householder's discussion (2) of the circuit composed of two neurons, a discussion which elaborates, somewhat, upon that of Rashevsky mentioned above. The terminology and theoretical assumptions are Rashevsky's.

Whereas Rashevsky and Householder considered only the case that each neuron develops a single substance, ε, we wish to consider also the development of ε and j, an excitatory and an inhibitory substance. For the self-exciting neuron, the one-substance problem leads

* In the figures we introduce a dot to represent the nucleus so that all impulses on one side of the nucleus travel toward it (dendrites), and on the other side away from it (axons or their collaterals). In complex circuits this convention may be especially useful.

to a single differential equation of the first order which is solvable by a quadrature.

We assume, then, with Rashevsky, that the neuron, upon application of a stimulus of intensity S at any dendrite, becomes excited in degree $E(S)$, where E is some function which is *monotonically increasing* and which is *zero unless* S *exceeds a certain threshold* value h. Thereupon the neuron develops the two substances ε and j at the extremity of the axon or of a collateral. The development of each is proportional to the degree of excitation $E(S)$, but each substance is dissipated at a rate proportional to its own concentration. The difference of these two substances, $\varepsilon - j$, acts as a stimulus of intensity $S = \varepsilon - j$ upon any neuron sending out a dendrite from the synapse at which the development takes place.

Now, suppose an outside stimulus acting for a time results in the development of ε and j at the synapse of the circuit until these have the values, say, ε_0 and j_0. If $\varepsilon_0 - j_0$ exceeds the threshold of the dendrite of the circuit, then re-excitation will be taking place. If the outside stimulus is now removed we shall have the equations

$$\begin{aligned} \frac{d\varepsilon}{dt} &= AE(\varepsilon - j) - a\varepsilon \,, \\ \frac{dj}{dt} &= BE(\varepsilon - j) - bj \,, \end{aligned} \tag{1}$$

governing the ensuing development, where $E(\varepsilon - j)$ denotes some function of $\varepsilon - j$. If the outside stimulus is not removed, however, we shall have to add to the first term on the right of each equation an additional positive quantity describing the continued action of this outside stimulus. If this added term in each equation is independent of ε and j, i.e., if the excitations resulting from stimulations of the two dendrites combine additively, then we may write, in place of (1), the equations

$$\begin{aligned} \frac{d\varepsilon}{dt} &= A[E_1 + E(\varepsilon - j)] - a\varepsilon \,, \\ \frac{dj}{dt} &= B[E_1 + E(\varepsilon - j)] - bj \,, \end{aligned} \tag{1'}$$

with E_1 a function only of this outside stimulus intensity.

We shall consider first the case in which the function E may be taken as linear with a sufficient degree of approximation, and second we shall consider the more general non-linear case.

2. The linear excitation function. We shall assume E_1 to be constant. Then E is proportional to $\varepsilon - j - h$, and if we absorb the constant of proportionality into A and B, we may write

$$E(\varepsilon - j) = \varepsilon - j - h. \tag{2}$$

Then the equations (1′) have the form

$$\begin{aligned} \frac{d\varepsilon}{dt} &= \alpha_1 + \beta_1\varepsilon - \gamma_1 j, \\ \frac{dj}{dt} &= \alpha_2 - \beta_2 j + \gamma_2\varepsilon, \end{aligned} \tag{3}$$

with

$$\begin{aligned} \alpha_1 &= A(E_1 - h), & \alpha_2 &= B(E_1 - h), \\ \beta_1 &= A - a, & \beta_2 &= B + b, \\ \gamma_1 &= A, & \gamma_2 &= B. \end{aligned} \tag{4}$$

Evidently γ_1, β_2 and γ_2 are positive; α_1 and α_2 have the same sign and are positive only when E_1, and hence the outside stimulus, is sufficiently large. The sign of β_1 depends upon the properties of the neuron.

If we set

$$\begin{aligned} \varepsilon_0 &= \frac{\alpha_1\beta_2 - \alpha_2\gamma_1}{\gamma_1\gamma_2 - \beta_1\beta_2}, \\ j_0 &= \frac{\alpha_1\gamma_2 - \alpha_2\beta_1}{\gamma_1\gamma_2 - \beta_1\beta_2}, \end{aligned} \tag{5}$$

or, employing the values (4),

$$\begin{aligned} \varepsilon_0 &= \frac{A(E_1 - h)}{a[1 - (\frac{A}{a} - \frac{B}{b})]}, \\ j_0 &= \frac{B(E_1 - h)}{b[1 - (\frac{A}{a} - \frac{B}{b})]}, \end{aligned} \tag{6}$$

then ε_0 and j_0 are the values of ε and j for which the right members of (3) vanish, so that the substitution

$$x = \varepsilon - \varepsilon_0, \qquad y = j - j_0 \tag{7}$$

gives

$$\frac{dx}{dt} = \beta_1 x - \gamma_1 y \,,$$

$$\frac{dy}{dt} = \gamma_2 x - \beta_2 y \,. \tag{8}$$

The characteristic equation of (8) is

$$\begin{vmatrix} \beta_1 - \lambda & -\gamma_1 \\ \gamma_2 & -\beta_2 - \lambda \end{vmatrix} \equiv \lambda^2 + (\beta_2 - \beta_1)\lambda + \gamma_1\gamma_2 - \beta_1\beta_2 = 0 \,. \tag{9}$$

We are interested in determining, if possible, values of the parameters which yield a stable equilibrium and those which yield periodic solutions. We shall take up first the case of the periodic solutions.

A system of linear differential equations possesses a pair of periodic solutions provided the characteristic equation has a pair of pure imaginary roots. Hence equations (8), and hence (3), have periodic solutions provided

$$\beta_2 - \beta_1 = 0 \,, \qquad \gamma_1\gamma_2 - \beta_1\beta_2 > 0 \,. \tag{10}$$

In terms of the original set of parameters, these conditions are equivalent to

$$A - B = a + b \,, \qquad 1 > \frac{A}{a} - \frac{B}{b} \,. \tag{11}$$

The first condition thus requires that A shall exceed B, i.e., that the rate of production of ε shall exceed the rate of production of j, by an amount which is dependent upon the rates of dissipation of ε and j, as well as upon the amounts of ε and j momentarily present. The second condition is certainly satisfied if the right member of the inequality is negative. A *simple* neuron of this type, one not forming a circuit, Rashevsky has called an inhibitory neuron of intermediate type, because if it is acted upon by a constant stimulus it develops first an excess of ε, but later an excess of j (**1**, pp. 224, 260-261).

These conditions are not sufficient, however, for actual periodicity. We must have, in addition, the conditions

$$\varepsilon_0 > 0 \,, \qquad j_0 > 0 \,,$$

since $(\varepsilon_0 \,, j_0)$ is the point about which the fluctuation occurs. If we compare (6) with the second condition (11), we obtain the added condition

$$E_1 > h \,, \tag{12}$$

and these conditions, (11) and (12), are necessary and sufficient for periodicity.

This last condition means that the circuit will not exhibit periodic fluctuations of excitation if left to itself, but only if a sufficiently large constant external stimulus is applied. Such a situation could arise if the circuit is connected with another circuit in dynamic equilibrium.

THEOREM 1. *A self-exciting neuron, developing both* ε *and* j *according to the linear differential equations of the form (3), can exhibit periodic fluctuations of excitation only when it is constantly excited by a sufficiently great external stimulus, never after it is left to itself. If it is stimulated from without by a constant stimulus great enough for (12) to be satisfied, then it will exhibit such periodicity provided its parameters satisfy the conditions (11), where* A *and* B *are parameters governing the rates of development of* ε *and* j*, respectively, and* a *and* b *are parameters governing the rates of their dissipation. Condition (11) concerns only the properties of the neuron; condition (12) involves the magnitude of the external stimulus.*

We turn next to the question as to the existence and stability of equilibria. There is always one and only one equilibrium point in the linear case, viz., the point (ε_0, j_0) defined by equations (5) or (6). But if this is to be a meaningful equilibrium, both coordinates, ε_0 and j_0, must be positive. It is evident that both will be positive or else neither will be positive.

Any equilibrium is stable only when the real parts of the roots of the characteristic equation are both negative (3). This requires first that the two roots shall be either complex, or else real and of like sign, and therefore that the second of conditions (11) shall be fulfilled. Hence the denominator in (6) must be positive and therefore, if the fractions are to be positive, the numerators must be positive and (12) must be satisfied. In place of the first of conditions (11), however, we have, as the condition that the real parts of the root shall be negative,

$$\beta_2 > \beta_1, \quad \text{or} \quad A - B < a + b. \tag{13}$$

Hence we conclude:

THEOREM 2. *A self-exciting neuron, developing both* ε *and* j *according to the linear differential equations of the form (3), can reach a state of steady excitation* (ε_0, j_0) *as given by (5) or (6) only when it is constantly excited by a sufficiently great external stimulus, never when left to itself. If it is stimulated from without by a constant stimulus great enough for (12) to be satisfied, then it will possess such a state of steady excitation provided its parameters satisfy (13) and the second condition (11). In these inequalities,* A *and* B *are*

parameters governing the rates of development of ε *and* j, *respectively, while* a *and* b *govern the rates of their dissipation. As in Theorem 1, condition (12) places a lower limit upon the intensity of the outside stimulus, while the other two conditions concern only the properties of the neuron itself.*

3. The non-linear excitation function. We now consider the case in which the excitation function $E(\varepsilon - j)$ is non-linear, and we assume that E is a *properly monotonically increasing* function of its argument $\varepsilon - j$, and that it possesses a *properly monotonically decreasing derivative.* Naturally we still assume that E is positive only when $\varepsilon - j$ exceeds a positive threshold value h. We consider the development only when the external stimulus is withdrawn, so that $E_1 = 0$. Hence we return to equations (1).

Equilibria occur at the solution (ε_0, j_0) of

$$E(\varepsilon_0 - j_0) = \frac{a\varepsilon_0}{A} = \frac{bj_0}{B}. \tag{14}$$

For these to be meaningful, each member of (14) must be positive. E is positive if $\varepsilon_0 - j_0 > h$. If we set

$$\varepsilon_0 = \frac{AS_0}{a(\frac{A}{a} - \frac{B}{b})}, \qquad j_0 = \frac{BS_0}{b(\frac{A}{a} - \frac{B}{b})}, \tag{15}$$

then

$$S_0 = \varepsilon_0 - j_0, \tag{16}$$

and equations (14) are equivalent to (15) and

$$E(S_0) = \frac{S_0}{\frac{A}{a} - \frac{B}{b}}. \tag{17}$$

Since the graph of E is increasing and concave downward, it is evident that (17) has not more than two real roots. In order that it may have two it is necessary that

$$\frac{A}{a} - \frac{B}{b} > 0, \qquad (\frac{A}{a} - \frac{B}{b})E'(\infty) < 1, \tag{18}$$

although these conditions are not sufficient.* We *assume,* therefore, that (17) has two distinct real roots, and hence that there are *two equilibria* besides the obvious equilibrium state of inactivity.

It is convenient to transform equations (1) into a single equation of the second order, by means of the substitution

$$S = \varepsilon - j\,. \tag{19}$$

If we differentiate equation (19) twice and equations (1) once we obtain, in all, seven equations involving S, ε, j, and their first and second derivatives. From these seven equations the six quantities, ε and j and their derivatives, can be eliminated algebraically, and when $a \neq b$ there results

$$\frac{d^2S}{dt^2} + [a + b - (A - B)E'(S)]\,\frac{dS}{dt} + abS - (bA - aB)E(S) = 0\,. \tag{20}$$

It is understood that $E'(S)$ signifies the derivative of E with respect to S.

Now let

$$S_0 = p\,, \qquad \text{and} \qquad S_0 = q > p \tag{21}$$

be the two roots of (17) and set

$$S = x + p \tag{22}$$

and

$$S = y + q\,. \tag{23}$$

The x and y used here are, of course, different from those of section 2. We shall show by the first transformation that p is always a point of unstable equilibrium. By the second transformation we shall show that under suitable circumstances the point q may be a point of either stable or else of periodic equilibrium.

To prove the first assertion, we set

$$(bA - aB)E(x + p) - abp = (bA - aB)\phi(x) \tag{24}$$

Then

$$E'(x + p) = \phi'(x), \tag{25}$$

and

$$\phi(0) = 0\,. \tag{26}$$

* A condition analogous to the second of (18) should have been added to those given in 2, p. 274, since failing such a condition only a single, unstable, equilibrium would exist. Physically we may expect even the stronger condition $E'(\infty) = 0$ because of the finite refractory period of the neuron (1, Ch. XXII).

Since (17) has no root smaller than p and one root greater, it follows that at p the derivative of the left member of (17) is greater than that of the right member. Hence, if we refer to (25) we see that

$$(\frac{A}{a} - \frac{B}{b})\phi'(0) > 1 . \tag{27}$$

Therefore if we write

$$\phi(x) = c_1^2 x - \cdots , \tag{28}$$

then (27) is equivalent to

$$(bA - aB)c_1^2 > ab . \tag{29}$$

Equation (20) is equivalent to

$$x'' + [a + b - (A - B)\phi'(x)]x' + abx - (bA - aB)\phi(x) = 0 . \tag{30}$$

If we retain only the linear terms in (30) as given by (28), we have, approximating x, the function ξ, where

$$\xi'' + [a + b - (A - B)c_1^2]\xi' + [ab - (bA - aB)c_1^2]\xi = 0 . \tag{31}$$

The coefficients of ξ and its derivatives in (31) are coefficients of the characteristic equation. But the inequality (29) shows that the constant term in the characteristic equation is negative, and therefore the characteristic roots have opposite signs. Hence the equilibrium is unstable.

Turning now to transformation (24), we set

$$(bA - aB)E(y + q) - abq = (bA - aB)\psi(y) . \tag{32}$$

Then

$$E'(y + q) = \psi'(y) , \tag{33}$$

and

$$\psi(0) = 0 . \tag{34}$$

Since q is the largest root of (17), if we write

$$\psi(y) = d_1^2 y - \cdots , \tag{35}$$

then we have

$$(bA - aB)d_1^2 < ab . \tag{36}$$

In place of (30) we have

$$y'' + [a + b - (A - B)\psi'(y)]y' + aby - (bA - aB)\psi(y) = 0 , \tag{37}$$

and in place of (31), the equation for the approximation η, to y,

$$\eta'' + [a + b - (A - B)d_1^2]\eta' + [ab - (bA - aB)d_1^2]\eta = 0. \tag{38}$$

The constant term of the characteristic equation here is positive, so that the characteristic roots are either complex, or else they are real and of like sign. For the equilibrium to be stable it is necessary and sufficient that we have

$$a + b > (A - B)d_1^2. \tag{39}$$

If the characteristic roots are complex, the approach to equilibrium (ε_0, j_0), given by (15) with q replacing S_0, is along a spiral path. If the roots are real, the approach is along one or the other of two determinate tangents. This may be compared to the two-neuron circuit, each neuron developing a *single* substance, ε, where the characteristic roots are always real and the approach to equilibrium always along a determinate tangent (2, p. 281).

For *periodicity* to arise, it is *necessary*, but not sufficient, that the inequality sign in (39) be replaced by equality:

$$a + b = (A - B)d_1^2. \tag{40}$$

This condition would be sufficient if the curvature of E were so small in the neighborhood of $S = q$ that the approximating linear equation (38) could be taken as exact. In general, however, even when (40) is satisfied so that first-order terms have no damping or dispersing effect (leading to the presence of the exponential factor in the solution multiplying the trigonometric terms) the terms of higher order may have such an effect, and for true periodicity an infinite number of conditions must be imposed. These are analogous to (40) but involve coefficients of terms of higher order in the expansion (35). However, this effect, in general, will be small.

In summarizing this case, therefore, we may state:

THEOREM 3. *Suppose the excitation function* E(S) *of the self-exciting neuron, which develops* ε *and* j *according to the differential equations (1), is properly monotonically increasing, has a properly monotonically decreasing derivative, and is such that equation (17) has two distinct real roots. Then equations (17) and (15) define two equilibria of the circuit when it is not acted upon by external stimulation, the equilibrium nearest the origin being always unstable. The other equilibrium will be stable, provided the parameters of the neuron satisfy condition (39). In case the more restrictive condition (40) is satisfied, the circuit may be periodic, although terms of higher order may exert damping or dispersing effects (which are, in general, small).*

A detailed analysis of the geometric form of the (ε, j), path of the state of the circuit can be made by the same method as was employed by Householder in discussing the two-neuron circuit (2).

4. A single-substance circuit. In deriving (20) it was necessary to assume that $a \neq b$. If $a = b$ we obtain only a first-order equation of the form

$$\frac{d\varepsilon}{dt} = A[E_1 + E(\varepsilon)] - a\varepsilon .$$

This is precisely the equation of the circuit formed by a neuron developing only ε, and no j. We consider this type of circuit in this section.

After making a change in time-units, if necessary, we obtain an equation of the same form as that given above, but in which $a = 1$ (2, p. 274). By making a change in the unit of the excitation intensity, also, we may make $A = 1$. We therefore assume both these changes to have been made, and we write

$$\frac{d\varepsilon}{dt} = E_1 + E(\varepsilon) - \varepsilon . \tag{41}$$

We consider only a constant E_1. Then equilibrium occurs at $\varepsilon = \varepsilon_0$ defined by

$$E_1 + E(\varepsilon_0) - \varepsilon_0 = 0 . \tag{42}$$

We assume again that the function E is properly monotonically increasing, that its derivative is properly monotonically decreasing, and that $E(h) = 0$. Then there are two possibilities, according to whether $E_1 > h$ or $E_1 < h$.

If $E_1 > h$, then (42) has a single root provided only

$$E'(\infty) < 1 . \tag{43}$$

Otherwise it has none. But if $E_1 < h$, then there are two real roots or else no real root when (43) is satisfied.

If $E_1 > h$ and there is a root of (42), then for $\varepsilon < \varepsilon_0$ the right member of (41) is positive, while for $\varepsilon > \varepsilon_0$ the right member is negative. Thus the equilibrium is necessarily stable. However if (43) is not satisfied, then the right member of (41) is positive for all values of ε, and the amount of ε increases to infinity so long as E_1 retains its constant value.

If $E_1 < h$ and (42) has two distinct roots, say $\varepsilon_0 = p$ and $\varepsilon_0 = q > p$, then

for	$h < \varepsilon < p$,	$\varepsilon' < 0$;
for	$p < \varepsilon < q$,	$\varepsilon' > 0$;
for	$\varepsilon > q$,	$\varepsilon' < 0$.

Hence $\varepsilon = q$ is a point of stable equilibrium, $\varepsilon = p$ is unstable. Again, if (43) were not satisfied, q would not exist, and if ε once exceeded p the production of ε would continue to infinity.

An increase in the value of E_1 results in an increase in the value of q and a decrease in or the disappearance of p if p exists initially. If the circuit is initially at rest, it can be started only by applying an external stimulus sufficient to make $E_1 > h$ and it must be continued until ε has time to exceed the value of p corresponding to $E_1 = 0$.

The time required for initiating this activity of the neuron is easily obtained from the solution of the differential equations. This solution can be written in the form

$$\int_{\varepsilon_1}^{\varepsilon} \frac{d\varepsilon}{E_1 + E(\varepsilon) - \varepsilon} = t - t_1 , \tag{44}$$

where ε_1 is the value of ε at the time t_1.

In partial summary we state the concluding theorem.

THEOREM 4. *Let a circuit be formed by a neuron developing only an excitatory substance ε according to the differential equation (41). Let the excitation function* $\mathrm{E}(\varepsilon)$ *be properly monotonically increasing with a properly monotonically decreasing derivative, and let it be such that equation (42) with* $\mathrm{E}_1 = 0$ *has two distinct real roots. Then for any value of* E_1, *the circuit has always a state of stable excitation. If* $\mathrm{E}_1 > \mathrm{h}$, *then this is the only equilibrium. If* $E_1 < \mathrm{h}$ *there is also an unstable equilibrium, the two equilibria being given by the two roots* p *and* $\mathrm{q} > \mathrm{p}$ *of (42). In this case the initially resting circuit can be brought to the state of activity by short application of an external stimulus sufficient to make* $\mathrm{E}_1 > \mathrm{h}$. *The time necessary for this application is given by the integral (44), where the lower limit ε_1 is zero and the upper limit ε is the value of* p *corresponding to the permanent value of* E_1.

5. Some other circuits. Circuits of apparently more complex form may, in special cases, be reducible to those previously discussed, and we shall conclude by noting a few examples.

The discussions of the two-neuron circuits which develop only the excitatory substance have dealt with the circuits only after the external stimulus has been withdrawn. When the external stimulus is acting, the equations for such a circuit have the form, analogous to equations (4) of (2),

$$\begin{aligned} \frac{d\varepsilon_1}{dt} &= \bar{E}_2 + E_2(\varepsilon_2) - \varepsilon_1 , \\ \frac{d\varepsilon_2}{dt} &= \bar{E}_1 + E_1(\varepsilon_1) - \varepsilon_2 , \end{aligned} \tag{45}$$

where the first neuron is stimulated by ε_2 and develops ε_1, and where $\bar{E}_1$ and $\bar{E}_2$ are the excitations produced by the external stimulus. If $\bar{E}_1$ and $\bar{E}_2$ are constant, then a simple linear substitution

$$\xi_1 = \varepsilon_1 - \bar{E}_2 , \qquad \xi_2 = \varepsilon_2 - \bar{E}_1 \tag{46}$$

replaces equations (45) by the equations

$$\begin{aligned} \frac{d\xi_1}{dt} &= H_2(\xi_2) - \xi_1 , \\ \frac{d\xi_2}{dt} &= H_1(\xi_1) - \xi_2 , \end{aligned} \tag{47}$$

where

$$H_1(\xi_1) = E_1(\xi_1 + \bar{E}_2) , \qquad H_2(\xi_2) = E_2(\xi_2 + \bar{E}_1) . \tag{48}$$

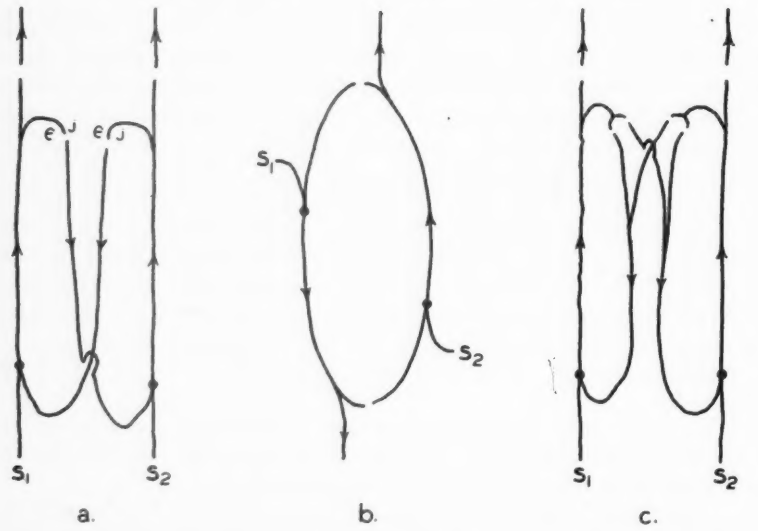

FIGURE 2

The functions H have the same monotonicity properties as the functions E, so that equations (47) are of the same form as the equations (4) of (2). The only difference is that the unstable equilibrium may no longer exist. The stable equilibrium remains.

Circuits of this type are represented by Fig. 2b, and, less obviously, by Fig. 2a. Here each fiber has a dendrite which reaches forward and synapses with a collateral to the axon of the other fiber.

Such a circuit as this, when it develops both substances, ε and j,

exhibits the same properties as the simple circuit formed by a single neuron provided the two fibers are identical, having the same excitation functions E, and the same parameters A, B, a and b, and provided, in addition, the external stimuli are always equal. For the two neurons will then be interchangeable, ε_1 and ε_2 will remain equal, j_1 and j_2 will remain equal, and we may therefore set $\varepsilon_1 = \varepsilon_2 = \varepsilon$, $j_1 = j_2 = j$, and write equation (1′) as governing the activity of this circuit. But if the neurons are not alike, or if the external stimuli are not the same, then the values of the (ε's) and those of the (j's) will not then be equal, and four equations are needed.

Even more complicated circuits can be reduced to this case when the component neurons are alike and the external stimuli equal. Another possible complication is illustrated in Fig. 2c, where a dendrite of each neuron branches and synapses with the axon collaterals of both neurons of the circuit. If the neurons are identical and the external stimuli equal, then the (ε's) are equal and the (j's) are equal as before. But since each neuron is stimulated by itself and also by the other neuron, the contribution of the circuit to the production of ε by either neuron is not AE, but $2AE$. Hence the equations are

$$\begin{aligned} \frac{d\varepsilon}{dt} &= AE_1 + 2AE(\varepsilon - j) - a\varepsilon\,, \\ \frac{dj}{dt} &= BE_1 + 2BE(\varepsilon - j) - bj\,. \end{aligned} \tag{49}$$

But these factors 2 do not alter the form of the equations, and if we replace, say, $2E$ by H, equations (49) are obviously of the form (1′). In fact, any number of identical neurons could be thus inter-connected by means of branching dendrites, and if the dendrites have the same number of branches the form of the equations is still the same.

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ALTERNATIVE VIEWS ON CORRELATIONS BETWEEN PERSONS

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The article discusses the general principles involved in factorizing correlations between persons, and compares the special techniques put forward by Burt and Stephenson respectively. The chief points of agreement and disagreement are summarized. In statistical procedure the two writers prove to be largely in accord; but they differ over the psychological and the methodological implications, the chief source of divergence being the so-called "reciprocity principle."

INTRODUCTION

During the past five years the authors of this joint paper have worked together in the same laboratory using the device of correlating persons, instead of correlating tests, as a basis for factor-analysis. Starting with virtually the same procedure, devised in the first instance for practical problems of educational and vocational guidance, they have found themselves in broad agreement in regard to the main principles and general conclusions. On the other hand, they have found themselves diverging more and more in regard to the theorectical implications of the method. Since in their own more recent publications it is naturally these subsequent divergences that have formed the centre of discussion, the numerous points of initial agreement have been overlooked by those who have taken an interest in similar problems.

The issues out of which the differences chiefly arise are seen most plainly in Burt's attempt to review the whole subject (8). His conclusions implied certain tacit assumptions which seemed so untenable to Stephenson that, on reading the draft of it in manuscript, he considered that less confusion would be caused if its publication were postponed until "a joint paper could be drawn up, either that we could all agree upon, or else giving our own versions of what we each

think we have done."* At the time this proved impracticable. The following summary, however, is a considered attempt to produce such an agreed statement. Our chief aim will be to formulate, as briefly and as concisely as possible, the main points both of agreement and of difference.

To begin with, we may perhaps indicate what we believe to be more or less original in our respective contributions. These happen to be the points to which our critics have mainly taken exception.

Initial Proposals

(A). Burt has made the following suggestions:

1. In obtaining trait-assessments for a number of persons, to begin by grading the various traits for one and the same person (or group of persons) instead of by grading the various persons for one and the same trait (e.g., in dealing with imagery, to inquire whether A's visual imagery is stronger than his other imagery, instead of inquiring whether A's visual imagery is stronger than B's), and to use these assessments for determining the mental "profiles" or "patterns" characteristic of different individuals, types, sexes, social classes, and the like. Such a method of grading has proved to be of special value (a) in practical work (educational, vocational, or clinical — cf. 2, 3, 4), particularly where the assessments have to be made, not by the same observer, but by different observers whose standards cannot well be equated; (b) in theoretical work (5, 9), particularly where it is the relative strength of all the traits rather than the absolute strength of each trait that provides the requisite description of the individual, type, or group.

2. On the basis of such gradings, to calculate covariances (or correlations) for persons as well as, or instead of, for traits.

3. To apply factor analysis to tables of covariances between persons as well as to tables of covariances between traits (8 and refs.).

4. More generally, to distinguish between row-factors, obtained by postmultiplying the measurement-matrix by its transpose and factorizing the product, and column-factors, obtained by premultiplying the measurement matrix by its transpose (e.g., by correlating rows and correlating columns, respectively). When the initial matrix is arranged according to the usual convention, the row-factors are factors for tests or traits, and the column-factors are factors for

* Eventually it was agreed that this particular paper, dealing with the alleged reciprocity between correlating persons and traits, should be published; and Stephenson's reply was added in a postscript to his article in this journal (7).

persons; but such designations are not universal, since in certain cases tests may become personified and appear as examiners, and other persons may be used as tests to test them (e.g., when the "reliability" of different interviewers is tested with a given batch of candidates); in other cases the columns may represent gradings obtained for one and the same person only, or for combined groups of persons.

5. To draw a sharp distinction in the results of factor-analysis (whether reached by correlating columns or rows, persons or traits) between (i) the first or general factor, which in nearly every form of factor-analysis (a) has positive saturations or loadings only, (b) emerges first (but may not be found at all), and (c) accounts for more of the total variance than any other; and (ii) subsequent or secondary factors, which, according to the method of analysis, will be either (a) "bipolar factors," i.e., factors having negative as well as positive saturations or loadings (as in multiple general factor analysis before rotation), or (b) "group factors," i.e., factors having mainly positive or (approximately) zero factor loadings (as in group-factor and bi-factor analysis, and in multiple factor analysis after rotation).

6. Adopting these distinctions, Burt claims to have shown (a) that the general factor for persons cannot be obtained by factorizing correlations between traits; (b) that the general factor for traits cannot be obtained by factorizing correlations between persons; (c) that with one and the same matrix, suitably standardized and suitably analyzed, the secondary bipolar factors are approximately the same whether obtained by correlating persons or correlating traits—exactly the same under certain definable conditions; (d) that secondary bipolar factors express differences between positive group factors (**9**, **10**).

7. To use the first or "general" factor for persons (which cannot be obtained by correlating traits) as a mode of measuring characteristics (e.g., correspondence with an ideal standard) which cannot be measured by other means. The weighted average of trait-measurements then provides the ideal standard — a set of "true" values by which the reliability of examiners or the preferences of judges may be assessed (cf. **11** and refs.). Where, however, the differences in weights are non-significant, Burt would prefer to use the method of analyzing variance, developed by Fisher.*

* In general the two methods lead to similar results, except in regard to tests of significance with small samples or non-linear correlations. When (as is commonly the practice in psychology) the trait-measurements are reduced to standard measure, the relations between correlational analysis and analysis of variance may at their simplest be summed up as follows: (i) the average intercorrelation is identical with the intra-class correlation; (ii) the average satura-

8. To use the secondary "bipolar" or "group" factors obtained by correlating persons (a) as a means of checking the alternative determination of types obtained by correlating traits, (b) as a method of determining types where the correlation of traits is impossible or unreliable (e.g., in the determination of imagery types based on self-assessments) (cf. 12 and refs.).

9. To regard the saturation coefficient for a person as itself a correlation with the type standard, and to use its inverse hyperbolic tangent (obtained by Fisher's z-transformation) to measure that person's approximation to the type. (The advantage of the transformation is that a normal distribution of the type-measurements is no longer precluded).

10. To base reliability coefficients for the same person on a number of trials instead of two, and by that means to study intra-personal variability and the "cyclic" or "spiral" character of such variations.

11. To use the relations of equivalence, obtaining between person-factors and trait-factors under certain restricted conditions, as a ground for selecting one particular system of factors from a large or infinite number of such systems that might otherwise be available (8).

For brevity, the adoption of these devices — particularly those described in paragraphs (1) to (9) — has occasionally been termed "*P*-technique" to distinguish it from the commoner methods of factor-analysis that start by correlating traits ("*T*-technique").

(B). Stephenson has made the following original suggestions:

1. To use two-factor theorems, instead of multiple factor theorems, for analyzing correlations between persons (6).

2. To distinguish between fractional and non-fractional factors, substituting the conception of fractional factors for that of multiple factors (13). Non-fractional factors refer to the total communality of a variable, and represent abilities and types; fractional factors seek to *analyze* them.

3. On this basis to develop correlating persons into a new technique ("*Q*-technique") leading to factors wholly independent of those that are obtained by correlating traits or tests. To this end he proposes to retain the fundamental distinction between populations and

tion coefficient is approximately equal to the correlation-ratio, calculated as the ratio of "variance between means" to "total variance"; (iii) in factor-analysis the basis of comparison is the mean of the squares of the saturation coefficients; in the analysis of variance it is virtually the square of the mean of the saturation coefficients (exactly so, with the summation method and perfect reliability; approximately so, with other methods or assumptions).

variables, and between the kinds of objectives that can be served by persons (a) as variables and (b) as items of a statistical population respectively (7).

4. To formulate a set of inverted theorems for such a technique* (6).

5. To introduce for these purposes new concepts such as "psychological significance" (distinct from "statistical significance"), "dominance," "fractional and non-fractional factors," etc.

6. To apply such methods and conceptions systematically (a) to the measurement of abilities by reference to a single standard person; (b) to evaluations obtained from a single person only; (c) to experimental work in the laboratory,† as distinct from educational or vocational work in the field, and so to solve problems of general as distinct from individual psychology (13, 14).

7. To construct on these foundations a new branch of typology, which may be termed "personalistics." For this purpose Stephenson proposes to distinguish between different uses of the word "type": (a) In one usage, types are merely arbitrary sections of a general tendency (e.g., introvert-extravert as extremes of a uni-dimensional scale of I - E); (b) in another, types are patterns of such tendencies (e.g., the common-sense type of person has more *w* than *g*): both of these pertain to types as "classes of beings having a common characteristic." (c) In the third usage, types are lived, specimen-entities — characteristic specimens of a special class of beings. Stephenson gives an explicit definition of this third usage, namely, any persons who satisfy the conditions for a non-fractional factor in *Q*-technique are by definition persons of a type (14).

8. Throughout, to distinguish rigidly between the methodological bases of *r*- and *Q*- techniques. Laws in *r*-technique derive from large numbers of cases (persons) and from individual differences; laws in *Q*-technique can derive from work on *one* person only, and are independent of individual differences.

Points of Agreement

After a good deal of mutual discussion we have eventually reached the following points of agreement. We both agree:

* Beebe-Center and Burt had deduced and used similar formulae for this purpose, but these were merely the old formulae for correlating traits applied to new problems, — Spearman formulae in the case of Beebe-Center, summation formulae in the case of Burt. As noted below, Burt regarded such formulae as capable of yielding first approximations only to the results obtained by a more complete multiple factor analysis based on covariances and the principle of least squares (5).

† Burt and Binns had applied the method to problems of touch discrimination, but only incidentally, not as a systematic branch of laboratory technique (3).

1. That correlating persons is a valid general statistical technique. In this we both differ from those who would use only product-moments or specially devised techniques (e.g., Rhodes) or who consider the whole procedure invalidated by incomparability of units (except in special and restricted cases, e.g., Thomson).

2. That for one and the same matrix the general factor for persons cannot be obtained by correlating traits, and similarly that the general factor for traits cannot be obtained by correlating persons.

3. That, in analyzing correlations between persons, instead of finding a single general factor sufficient to explain all the correlations (as Beebe-Center has apparently maintained), secondary or "group" factors are almost invariably discoverable. Thus, in analyzing correlations between persons as in analyzing correlations between traits, we both now agree that it may be necessary to envisage factors of four types: (i) general factors, (ii) group factors, (iii) specific factors, (iv) error factors (5, p. 259).

4. That, in analyzing correlations between persons, besides general factor techniques, what Burt has called the group-factor technique and Holzinger the bi-factor technique is therefore permissible.*

5. That the saturation coefficients obtained by correlating persons may be used to measure the person's approximation to a type—the type being regarded as a single ideal (or idealized) person.

6. That in all research involving factor-analysis the choice of an appropriate experimental design is quite as important as the choice of an appropriate method of mathematical analysis, and that the factors discovered must depend quite as much upon the former as upon the latter.

Points of Difference

On the following points we find ourselves in sharp disagreement. We therefore put them forward as indicating a program for future discussion and research.

1. We differ, first of all, over the importance of the differences. Stephenson regards them as wide and fundamental; Burt thinks them small and subordinate, and capable of ultimate reconciliation.

2. The general character of the differences may be summed up by saying that Stephenson insists on a sharp opposition between *r*-technique and *Q*-technique, whereas Burt would regard them as involving much the same aims, methods, and theorems, and, in principle, though not always in fact, as merely alternative ways of analyzing any rectangular table of figures.

* Stephenson would point out that Spearman workers have long made analyses substantially similar to that recently described by Holzinger as bi-factor.

3. The differences seem mainly to spring from a fundamental difference of aim: this appears to have been overlooked by our earlier critics who noted only the superficial similarity in procedure. In substituting correlations between persons for correlations between traits, Burt remained preoccupied with much the same problems as before, and consequently (quite wrongly in Stephenson's opinion) treated the adoption of the alternative approach mainly as a matter of practical necessity or convenience. Stephenson's *Q*-technique is concerned with an entirely new set of problems, and is therefore put forward with objectives wholly different from those envisaged by *r*- or *P*-technique.

4. We differ over our general methodological approach. For elementary statistical reasons, Stephenson draws a sharp distinction between statistical variables and statistical populations, which Burt appears to confuse: *r*-technique applies to a population of persons with tests or traits, etc., as variables; *Q*-technique to a population of tests or traits, etc., with persons as variables. By retaining this distinction Stephenson seeks to remind the factorist that it is his duty, as a psychologist, to propose psychological hypotheses, and to "control" these by appropriately conceived variables. He therefore refuses to reduce everything to statistical populations both ways in any one matrix.

Burt does not deny the distinction between variables and populations (or rather constants descriptive of total populations); but he regards it is an *ad hoc* rather than as an absolute distinction: he sees no *a priori* reason why the constants in one mathematical argument should not become the variables in another. For him the term variable simply denotes any one element in a class or "population." In factor analysis we are concerned with three "populations" or classes, namely, "persons," "traits," and "factors"; and, in representing their relations geometrically, we may, according to convenience, take any one of the three as the initial axes of reference, thus obtaining "*P*-axes," "*T*-axes," or "*F*-axes." Since, as a rule, the factors are by definition independent, Burt regards it as more convenient (but not essential) to treat the rectangular *F*-axes as fixed (though initially unknown), and, according to the nature of the problem or the data, to represent either traits or persons by vectors in the space defined by these factors.

5. Stephenson holds that his own conception of *Q*-technique involves a complete break with the concepts of *r*-technique, i.e., the endeavour to describe psychological phenomena in terms of mental "abilities" and mental "tendencies" as these are understood in *r*-technique. In his view Burt's statements about the limitations of correlating persons show that he still retains the standpoint of earlier psy-

chometry, i.e., he thinks exclusively in terms of the "abilities" or "tendencies" of the old *r*-technique. His work on *P*-technique, therefore, appeared to Stephenson to have entirely missed the essential possibilities implicit in the idea of correlating persons. Consequently, though suggested in the first instance by issues raised at the International Conference on the analysis of examination marks and at a laboratory discussion on the applicability of Burt's technique to such problems (June, 1935), Stephenson's developments of the method must be regarded as the independent outcome of special problems of his own. He maintains that, though person-factors had previously been used for problems conceived on *r*-technique lines, the wider implications of such factors had been, and still are, almost entirely ignored, except in work based explicitly on his own conception of *Q*-technique. As a consequence, all earlier investigations, based on correlating persons or extracting person-factors, appear to him to come far short of a general psychometric technique avowedly defined as such, for the plain reason that it has refused to break (or has never thought of breaking) with conceptions that are part and parcel of *r*-technique.

6. We thus differ (at least in Stephenson's view) in regard to the traits, tests, or type of marking with which the investigations should start. According to Stephenson, in *Q*-technique they must be different from those used in *r*-technique. The view put forward by Burt and accepted by Vernon, namely, that the self-same traits that are used as variables in correlating traits can change in the twinkling of an eye into chameleon-like items of a statistical population when correlating persons, appears to Stephenson a gratuitous assumption. *Q*-technique thus leads to entirely new principles of test construction.

7. We apparently differ in regard to the units to be used in correlating persons. In measuring the traits Burt would prefer (a) to work with absolute units where possible, making no assumption as to the distribution of persons or traits; failing absolute units, (b) to start with measurements first standardized for traits or tests; (c) where both are impossible or inappropriate, to select the traits or test items so that they conform as closely as possible to a specified distribution, e.g., rectilinear or normal. In Stephenson's view this limits the units to those that pertain to, and are defined in terms of, individual differences. In his *Q*-technique individual differences are wholly irrelevant; a new unit of quantification has therefore to be forged which may be termed "significance." The preliminary standardization for traits relative to a population of persons, he believes, can lead only to trivial results; and in *Q*-technique he would begin by standardizing measurements for persons on the assumption of a normal distribution: i.e., for *Q*-technique the units must be so chosen

that their mean must be zero and the standard deviation unity for the population of traits within the person.

8. Where absolute units are available, Burt proposes to base the analysis on product-moments (or on averaged product-moments, i.e., variances and covariances) rather than on correlations. He admits that for many psychological measurements this may at present be impracticable; and he agrees that correlation is the better method where the essential point is to measure the resemblances between persons rather than to analyze variance. Stephenson, on the other hand, considers that correlation should *always* be used, since (a) covariance foregoes the all-important distinction between variables and statistical population and (b) variances in psychology can be made arbitrarily just what we please.

9. For the same reason Stephenson holds that the "analysis of variance" (as developed by Fisher and Snedecor) is inapplicable to psychological measurements. Burt, on the other hand, regards the applications here proposed [page 271 above, (A), para. 7] as a natural extension of the early procedure adopted by Cattell (1) and his followers; and he considers that by using the analysis of variance instead of factor-analysis more accurate tests of significance are made available, particularly when the samples are small, the correlations non-linear, and only one or two "factors" are at issue.

10. We differ fundamentally over the type of factor-analysis to be employed. Burt's original memorandum on the subject assumed that a *multiple-factor* technique alone was appropriate. Stephenson, on the other hand, in his very first contribution on correlating persons, put forward a *two-factor* technique instead, and has worked out afresh a set of theorems to this end: he maintains that, in correlating persons, the two-factor technique is not only sufficient, but closer to psychological needs. Burt has occasionally adopted this simpler technique in early work where the correlations between persons appeared to present a single-factor hierarchy (e.g., 2, p. 136), but considers it invalid as a general rule, since with low probable errors more than one common factor is nearly always demonstrably required.

11. We differ somewhat in the considerations which make us prefer in certain cases to use the bi-factor, group-factor, or submatrix technique (Burt's method *a*, 5, p. 306). Burt believes that it is appropriate only where the variables form discontinuous groups (10). Stephenson, on the other hand, would here distinguish between non-fractional and fractional factors (which will usually be oblique). To discover types, he merely reorganizes a table of correlations into clusters (method *a*) each of which is sufficient and necessary to define a type (13).

12. Our chief differences, however, turn on the relations obtaining between test-factors and person-factors. Burt believes that, where conditions are comparable, the factors obtained by *r*-technique and *Q*-technique are virtually the same, except, of course, the first or general factors. In the simplest case (covariating rows or columns of the same doubly-centered matrix) he maintains that the factor-loadings for persons are mathematically identical with the factor-measurements for traits, and the factor-measurements for persons with the factor-loadings for traits; under less rigid conditions he regards them as differing only in respect of minor inequalities in weighting (8). Stephenson holds that Burt's algebraic proof is vitiated by untenable assumptions, and, further, that, even if it were true, the deductions drawn would render the whole procedure useless, since it would follow that it was waste of time to correlate persons as well.

13. As a consequence of this so-called "reciprocity principle" (15), Burt holds that, in theory, secondary or group-factors obtained in correlating persons should be of the same nature and number as in correlating traits: in practice, since factors in the first instance merely define principles of classification, not concrete mental entities, we are free to take as few or as many of this number as we please. Stephenson regards the factors obtained in *r*-technique as defining the fundamental abilities or tendencies of men: consequently, in correlating tests the factors will be narrow and rare. On the other hand, in *Q*-technique we seek to map out the field into groups of persons who resemble one another with respect to whole aspects of their personality: consequently, in correlating persons the factors will be numerous and broad. After all, in any table of measurements the number of tests is comparatively small, while the number of persons may run into hundreds or thousands. To this Burt would reply that, although the *order* of the correlation-matrix for persons may as a result be much greater than the *order* of the correlation-matrix for tests, the maximum *rank* of either matrix is limited by the smaller number of the two. Hence the maximum number of independent factors (and, he would maintain, the actual number of significant factors) must remain the same in either case. Stephenson holds that the number of factors is a question of fact, not of mathematical parsimony or practical convenience.

14. Stephenson would add that Burt's arguments confuse a reciprocity that may conceivably apply to one and the same set of data with Stephenson's postulation of two distinct sets of data that are quite unrelated and serve different psychological objectives: *r*-technique pertains to individual differences, *Q*-technique pertains to general psychology, and is in no way concerned with individual differ-

ences (cf. 7 and 13, p. 9). Burt replies that on Stephenson's own showing each of these two sets can still be analyzed both ways,* and, in point of fact, all Stephenson's own analyses by *Q*-technique seem so far to have led to almost exactly the same factors as had previously been obtained by *r*-technique (linguistic and non-linguistic types, extraverted and introverted types, objective and subjective types, etc.).

15. We differ over the distribution of trait-measurements for one and the same person. Stephenson postulates a distribution fitting the normal curve of error (13, pp. 191, 193). Burt would agree that traits and test-material *can* be selected so as to fit, more or less reasonably, a normal distribution; but, as with the distribution of populations of persons for a single trait, so with the distribution of "populations" of traits for single persons, he considers that the "postulate" of normality should be accepted only after it has been empirically confirmed. Actually the lists of traits commonly employed (e.g., 13, p. 20) appear to him to be not only non-normal, but discontinuous and multi-modal.

16. We therefore differ widely over the occasions on which we believe correlating persons to be legitimate. Burt holds that, if the traits are normally distributed, correlation may be used; alternatively, if the units of measurement are objective, covariance may be used. But, since neither condition is easily obtainable, he would prefer, where possible, to base the analysis of types on correlations between traits rather than on correlations between persons. Since *Q*-technique aims at entirely different types, Stephenson cannot accept this suggestion.

17. We differ over the distribution of the types measured by the saturation coefficient for persons. Here Burt believes that the distribution of factor saturations, obtained by correlating persons and transformed to *z*-values, is approximately (though not exactly) normal. Stephenson believes that the results indicate separate and discontinuous types — as distinct as primroses from buttercups — so that certain persons may have zero saturations for certain type-factors.

18. We differ over the meaning to be attached to the factor-measurements for the type. We agree that the saturation coefficient for a person expresses the extent to which he fits this pattern. But Burt regards the type described by this pattern as an extreme and hypothetical ideal, towards which actual persons more or less approxi-

* "Nor is any analysis of data complete until it is examined both ways, by *r*- and *Q*-techniques" (6, p. 360).

mate, but with which they never perfectly correlate. Stephenson regards such a pattern as a caricature. For him the ideal representative of the type is a specimen personality, i.e., the *modal* person, not the extreme of a normal (or nearly normal) distribution (13, p. 193).

19. These further divergences are evidently related to the particular interpretations we attach to the notion of psychological "types." For Burt a type is merely a pattern of weighted assessments, into which different individuals fit more or less closely. As such it is expressible in terms of a single factor. All types are therefore related to corresponding tendencies, which may be highly complex; but such tendencies are descriptive only. It follows that we can distinguish, if we wish, as many types (or pairs of types) as there are factors discoverable in correlating traits. To this Stephenson can agree only with important reservations. For him, a psychological type [as defined in paragraph 7 (c) above] has no *mathematical* relation to the tendencies or abilities which are the concern of r-technique alone. Types which are the concern of Q-technique are related to such underlying tendencies in a *statistical* fashion only: between the underlying tendencies and the lived personality the relation is so involved and highly ramified that statistical representation alone does justice to the facts. Believing, as he does, that only about five common factors can be discovered when correlating traits, he cannot admit that the number of types is limited by the number of factors discoverable by r-technique. On the other hand, when correlating persons, just as he believes in an innumerable number of factors, so he believes in an innumerable number of types: types, in fact, as he defines them, can be as numerous as the genera and species of animals in the present epoch of fauna.

20. Thus, although expressed in statistical terms and producing minor differences in statistical procedure, the fundamental differences between us are not so much statistical as methodological. In Q-technique there is to Stephenson's mind nothing *statistically* novel, and hence statistical criticisms of this formula or that are of little moment. In Burt's work on person factors there is, he admits, statistical novelty, but he regards it as psychologically unimportant. On the other hand, Stephenson believes that on the *psychological* side Q-technique opens an entirely new field, namely, typology as he has defined it above: and it is this, not the mere mathematical procedure, that he has throughout desired to stress.

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STUDIES IN MATHEMATICAL THEORY OF HUMAN RELATIONS. II

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In continuation of previous studies, relations are studied between two classes of population, of which one is characterized by a much greater ability to organize and supervise the productive activities of the other. Under some special and rather simple assumptions, it is shown that this kind of interreaction results first in an increase of the ratio of the cumulative results of productive activities for the two classes in favor of the first. With time, however, this ratio reaches a maximum and declines. An expression for the "life span" of such organized classes is obtained.

In the second section, a study is made of possible non-uniformities of spatial distribution of the population. Some possible applications are made to the theory of the variation of the ratio of urban to rural population, showing how that ratio may increase with increasing total population.

I

In a previous paper (1) we discussed some possible interrelations between social classes based on the different amounts of initiative and imitativeness. In the present paper we shall discuss an interreaction of classes based on a different characteristic, namely, the difference in different people of the ability to perform different kinds of work and to produce different goods. In a sense, such discussion should lead us into the domain of mathematical economics, and in this field a very considerable progress already has been made by well-known works (2) of mathematical economists. However, we shall approach these problems from a somewhat different point of view, which will bring out particularly the social aspect of the interrelation of economic classes. In order to do so, we shall start again with highly oversimplified hypothetical cases. In view of the fact that much more complex and realistic cases have already been treated by the above-mentioned scientists, it may seem rather strange to start again with oversimplified cases. However, as we shall see, the problem which interests us here is different from the usual type of problems treated, offering its own complications; this consideration justifies the apparent set-back. Our failure to make use of those well-known results is due not to ignorance or neglect, but to the circumstance that the particular problems discussed here are not yet ripe for the use

of those more advanced results. This will become apparent to the reader as he peruses section I. (Cf. also the concluding paragraph of section I.)

Let us again consider for simplicity a society composed of two types of individuals *I* and *II*, their corresponding numbers being N_1 and N_2. Let these two types of individuals form two corresponding social classes, class *I*, and class *II*. Let us consider the following purely hypothetical case.

Each individual of type *I* produces per unit time an amount p_1 of goods necessary for the maintenance of his life, and he consumes per unit of time an amount c_1 of these goods. Similarly every individual of type *II* produces and consumes correspondingly amounts p_2 and c_2. Let

$$N_1 << N_2 \; ; \varepsilon_1 = p_1 - c_1 > 0 \quad \text{and} \quad \varepsilon_2 = p_2 - c_2 < 0 . \qquad (1)$$

Then, in the absence of any exchange of external supply, the individuals of type *I* are capable of indefinite existence, those of type *II* are not.

Now let the individuals of type *I* be capable of *organizing* their work, as well as the work of individuals of type *II*, in such a way that the production of the necessary goods is largely increased. In other words, let the production of the goods require a certain amount of *organization skill*, which is possessed by individuals of type *I*, but not by those of type *II*, so that while for themselves the individuals of type *II* cannot produce enough goods, they can do so under the direction of the individuals of type *I*.

While this is a rather abstract situation, yet it reminds one of some real situations. Thus a large number of workers, who produce complicated machinery in a factory as a result of work directed by a group of engineers and executives, would not be able to produce the same results, when left to themselves.

Under the conditions assumed above, the following relation will naturally establish itself between the individuals of type *I* and those of type *II*. The latter will agree to work under the direction of the former, so as to increase their output in goods, provided they do receive a part of the goods thus produced for their consumption. On the other hand, the individuals of type *I*, unless they act purely altruistically, a supposition too unlikely to be considered even theoretically, will agree to direct the work of the type *II* individuals only if they receive also part of the goods thus produced. Each individual of type *II* will give an amount w of his labor per unit time. The amount w may be conveniently measured in working hours per day. In compensation for that amount of work, he will get an amount θw of goods.

θ may be considered as the price of his labor, paid not in money, but in goods directly.

If, for simplicity again, we consider the case in which all N_2 individuals of type *II* are giving their work, the total amount P of goods produced *under the direction* of individuals of type *I* will be proportional to $N_2 w$, thus

$$P = AN_2 w . \tag{2}$$

The coefficient of proportionality A will itself depend on both N_1 and N_2 in general, so that

$$A = A(N_1, N_2). \tag{3}$$

The nature of this functional dependence will itself be determined by the character of the labor involved, method of production, etc. In the simplest case, however, it can be seen that $A(N_1, N_2)$ should, at least approximately, be a function of N_1/N_2, so that

$$A = f(\frac{N_1}{N_2}) > 0 . \tag{4}$$

The reason for this assertion will be seen, if we remember that, other conditions being kept constant, increasing both N_1 and N_2 m times should approximately increase P also m times. Hence such an increase must leave A invariant.

Some other properties of the function f in (4) may be found from general considerations. Since in the absence of individuals of type *I* no goods are produced *under their direction*, we must have

$$f(0) = 0 . \tag{5}$$

Keeping N_2, the number of workers available, constant but increasing the ratio

$$\eta = \frac{N_1}{N_2} , \tag{6}$$

by increasing N_1, will, for small values of N_1, increase P. But, when for a constant N_2, N_1 exceeds a certain amount, any further increase of N_1 will not result in any increase of P. A certain number of workers may require a definite number of supervisors, but an excessive number of supervisors does not increase the output. Hence $f(\eta)$ must either have a maximum for a certain value of η, or it must increase with η only up to a certain value of η, and for any further increase of the latter remain practically constant.

In order to obtain closed expressions, we may consider as a very rough approximation

$$f(\eta) = f_0(1 - e^{-a\eta}) . \tag{7}$$

This of course does not preclude consideration of other forms. We may take also

$$f(\eta) = a_1\eta - a_2\eta^2 \tag{8}$$

and consider it only in the range $0 < \eta < a_1/a_2$, for which the expression (8) is positive. It has a maximum for $\eta = \frac{1}{2}(a_1/a_2)$, the maximum value being $f_m = \frac{1}{4}(a_1^2/a_2)$.

Introducing (6) into (4) and then the latter into (2), we have

$$P = f(\eta)N_2w . \tag{9}$$

Of the total amount P of goods produced, an amount $\theta N_2 w$ is received by the N_2 individuals of the type II, the remaining amount being retained by those of type I. Hence the net rate of increase of the total amount W_1 of goods, produced by individuals of type I themselves and by those of type II for the class I, is:

$$\frac{dW_1}{dt} = N_1\varepsilon_1 + f(\eta)N_2w - \theta N_2w ; \tag{10}$$

and, similarly for the class II,

$$\frac{dW_2}{dt} = N_2\varepsilon_2 + \theta N_2w . \tag{11}$$

Even though, according to equ. (1), $\varepsilon_2 < 0$, dW_2/dt may be positive, since $\theta N_2 w > 0$. It must be remarked however, that (1) is not necessary at all in order to have (10) and (11). Quite generally, we obtain the same type of interreaction of the two classes even when both ε_1 and ε_2 are positive. This type of cooperative interaction may increase the rates dW_1/dt and dW_2/dt from their values $N_1\varepsilon_1$ and $N_2\varepsilon_2$ which they would have in the absence of such a cooperation. This presupposes such a choice of constants that

$$f(\eta) > \theta . \tag{12}$$

Inequality (12) fixes the highest "price" that the first class will be willing to pay for labor. The actual θ is determined by the demand of the individuals of type II for the goods produced. Just as in the standard treatments of mathematical economics, the larger θ per unit of w, the more of w an individual is willing to give. Hence

$$w = u(\theta) , \tag{13}$$

u being an increasing function of θ. For simplicity let us put

$$w = w_0 - \frac{b}{\theta}\,, \tag{14}$$

w_0 and b being constants. Expression (14) is suggested by the use of simple linear approximation for demand curves and by remembering that $\frac{1}{\theta}$ may be considered as the "price" of the goods in terms of labor. For $\theta < b/w_0$, individuals of type *II* will not agree to work at all. For $\theta = \infty$ they cannot work more than the physically possible amount w_0.

Introducing (14) into (10) gives:

$$\frac{dW_1}{dt} = N_1\varepsilon_1 + N_2[f(\eta) - \theta]\,(w_0 - \frac{b}{\theta})\,. \tag{15}$$

dW_1/dt is now a function of θ, and if the individuals of type *I* wish to make dW_1/dt as large as possible, they will agree on a θ which makes the right-hand side of (15) a maximum. This happens for

$$\theta = \sqrt{\frac{bf(\eta)}{w_0}}\,, \tag{16}$$

and, for the maximum value of the right-hand side of (15), the latter becomes

$$\frac{dW_1}{dt} = N_1\varepsilon_1 + N_2[w_0 f(\eta) + b - 2\sqrt{bw_0 f(\eta)}]\,. \tag{17}$$

Introducing (16) into (14), and then both into (11), we find:

$$\frac{dW_2}{dt} = N_2[\varepsilon_2 - b + \sqrt{bw_0 f(\eta)}]\,. \tag{18}$$

Equ. (17) may be written thus:

$$\frac{dW_1}{dt} = N_2[\varepsilon_1\eta + w_0 f(\eta) + b - 2\sqrt{bw_0 f(\eta)}]\,. \tag{19}$$

In order for any such interreaction as discussed here to be possible at all, both classes must gain something. This imposes certain inequalities on the constants, namely,

$$\varepsilon_1\eta + w_0 f(\eta) + b - 2\sqrt{bw_0 f(\eta)} > 0 \tag{20}$$

and

$$\varepsilon_2 - b + \sqrt{bw_0 f(\eta)} > 0\,. \tag{21}$$

If (21) is to hold, and if $\varepsilon_2 < 0$, then a necessary condition is

$$\sqrt{bw_0f(\eta)} - b > 0 .$$

Because of (16), and since necessarily $\theta > 0$, this requires:

$$b < w_0f(\eta) . \tag{22}$$

But (22) is not sufficient in this case. In the interval $0 < b < w_0f(\eta)$, the expression $\sqrt{bw_0f(\eta)} - b$ is positive and has a maximum for

$$b = b_m = \frac{w_0f(\eta)}{4} , \tag{23}$$

the maximum value being equal to

$$\frac{w_0f(\eta)}{4} . \tag{24}$$

Hence, for $\varepsilon_2 < 0$, the inequality (21) requires not only inequality (22) but also

$$\frac{w_0f(\eta)}{4} > |\varepsilon_2| . \tag{25}$$

Requirement (20) may be replaced by a more rigid one, namely

$$w_0f(\eta) + b - 2\sqrt{bw_0f(\eta)} > 0 . \tag{26}$$

Only in this case is dW_1/dt larger with the cooperative arrangement than without it. But for values of b in the neighborhood of b_m the expression $b - 2\sqrt{bw_0f(\eta)}$ is negative. However, for $b = b_m$ this expression equals $-\frac{3}{4}\, w_0f(\eta)$. Hence for $b = b_m$ the left-hand side of (26) equals $\frac{1}{4}w_0f(\eta)$, and (26) is therefore satisfied.

These considerations show that b would have to be in the neighborhood of b_m in order that there would be a gain from cooperation to individuals of class *II* for $\varepsilon_2 < 0$. A value of b too large or too small results in a loss to the individuals of type *II*. If, however, $\varepsilon_2 > 0$, then all that is necessary is that b should be sufficiently small. In this case too large a "price" asked for labor will result in the impossibility of reaching any agreement or cooperation.

If $\varepsilon_2 < 0$ and (25) holds, then with $b \approx b_m$, $dW_1/dt > dW_2/dt$. Equations (19) and (18) integrated give:

$$W_1 = N_2[\varepsilon_1\eta + w_0f(\eta) + b - 2\sqrt{bw_0f(\eta)}]t + W_{01} \tag{27}$$

$$W_2 = N_2[\varepsilon_2 - b + \sqrt{bw_0f(\eta)}]\, t + W_{02} , \tag{28}$$

where W_{01} and W_{02} are initial values. If we start with equal amounts of W per capita, so that

$$\frac{W_{01}}{W_{02}} = \frac{N_1}{N_2} \tag{29}$$

or

$$W_{01} = \eta W_{02} , \tag{30}$$

then for $t = 0$, $W_1/W_2 = \eta$, while for sufficiently large t this ratio tends to

$$\frac{W_1}{W_2} = \frac{\varepsilon_1 \eta + w_0 f(\eta) + b - 2\sqrt{b w_0 f(\eta)}}{\varepsilon_2 - b + \sqrt{b w_0 f(\eta)}} > 1 > \eta . \tag{31}$$

Thus there will be a gradual increase of the ratio of accumulated goods for the two classes.

However, things will happen in this manner only when N_1 and N_2 remain constant. The situation will be different if we consider a sufficiently long period of time, during which not only the size, but also the composition of the classes changes. Originally we have N_1 individuals of type *I* forming class *I*, and $N_2 > N_1$ individuals of type *II* forming class *II*. Let us consider again, as in the previous paper **(1)** that the progeny of parents of a given class continue to belong nominally to the same class, although they may be of a different type. Denoting by n_1' the number of individuals of type *I* in class *I*, by n_1'' the number of individuals of type *II* in class *I*, by n_2' the number of individuals of type *I* in class *II*, and by n_2'' the number of individuals of type *II* in class *II*, let us put:

$$\begin{aligned} n_1' + n_1'' &= n_1 ; \\ n_2' + n_2'' &= n_2 . \end{aligned} \tag{32}$$

Denoting by β the rate of increase of the population, assumed the same for all types, and by α the fraction of individuals of type *I* born, we have **(1)**:

$$\frac{dn_1}{dt} = \beta n_1 \quad ; \quad \frac{dn_2}{dt} = \beta n_2 ; \tag{33}$$

$$n_1 = n_{01} e^{\beta t} \quad ; \quad n_2 = n_{02} e^{\beta t} ; \tag{34}$$

and

$$\begin{aligned} \frac{dn_1'}{dt} &= \alpha\beta n_1 \quad ; \quad \frac{dn_1''}{dt} = (1-\alpha)\beta n_1 ; \\ \frac{dn_2'}{dt} &= \alpha\beta n_2 \quad ; \quad \frac{dn_2''}{dt} = (1-\alpha)\beta n_2 . \end{aligned} \tag{35}$$

Remembering that for $t = 0$, $n_1' = n_{01}$, $n_1'' = 0$; $n_2' = 0$; $n_2'' = n_{02}$, we find from (35) and (34)

$$n_1' = \alpha n_{01}(e^{\beta t} - 1) + n_{01} \quad ; \quad n_1'' = (1 - \alpha) n_{01}(e^{\beta t} - 1) \; ;$$
$$n_2' = \alpha n_{02}(e^{\beta t} - 1) \quad ; \quad n_2'' = (1 - \alpha) n_{02} e^{\beta t} + \alpha n_{02} . \tag{36}$$

For simplicity we neglect here the effect of "vertical mobility" from class to class (1, 4).

According to the picture made here, these variations of the different n_k^i result in a variation of η, for η is nothing else but n_1'/n_2. The quantities ε_1 and ε_2 are also affected by these changes, for when the whole class I of $n_1 = n_1' + n_1''$ individuals is not composed of individuals of type I, but of individuals of both types, we should substitute for ε_1 and ε_2 average values, such as

$$\bar{\varepsilon}_1 = \frac{n_1'\varepsilon_1 + n_1''\varepsilon_2}{n_1} \quad ; \quad \bar{\varepsilon}_2 = \frac{n_2'\varepsilon_1 + n_2''\varepsilon_2}{n_2} . \tag{37}$$

Putting

$$\eta = \frac{n_1'}{n_2} \quad , \quad N_2 = n_2 , \tag{38}$$

introducing (36) into (37) and (38), and then introducing the result into (19) and (18), we obtain differential equations for W_1 and W_2, which, however, are too cumbersome to be treated directly. In view of the schematizations already introduced, an exact solution of these equations would hardly be worth while. However, we may derive approximately some general properties of the solutions, without actually solving the equations.

Let us again consider the ratio W_1/W_2. As we have seen, it originally increases, tending to a constant value given by (31). This value, however, remains constant only as long as η, ε_1 and ε_2 are constant. Since this is not the case, W_1/W_2 will change. From (34) and (36) it follows that n_1/n_2 remains constant, but that n_1'/n_1 decreases. Equation (37) shows that $\bar{\varepsilon}_1$ decreases from $\bar{\varepsilon}_1 = \varepsilon_1$ to $\bar{\varepsilon}_1 = \varepsilon_2 < \varepsilon_1$; while $\bar{\varepsilon}_2$ increases. The quantity η decreases, because n_1'/n_2 decreases. This will result in a decrease of the numerator of (31) and an increase of the denominator, hence in a decrease of W_1/W_2. We have specifically considered the situation in which originally $dW_1/dt > dW_2/dt$, in other words, when [cf. equ. (18) and (19)]:

$$\varepsilon_1\eta + w_0 f(\eta) + b - 2\sqrt{b w_0 f(\eta)} > \varepsilon_2 - b + \sqrt{b w_0 f(\eta)} . \tag{39}$$

We shall investigate how inequality (39) is affected by a variation of

η. In order to discuss this, we must make same assumption about $f(\eta)$. If $f(\eta)$ is of the form (7), then very roughly we may approximate it for small values of η by

$$f = f_0 a\eta \tag{40}$$

in the range from $\eta = 0$ to $\eta = 1/a$, and by

$$f = f_0 \tag{41}$$

for $\eta > 1/a$. In other words, we substitute instead of the curve (Fig. 1) two segments of straight lines. Since a variation of η will affect

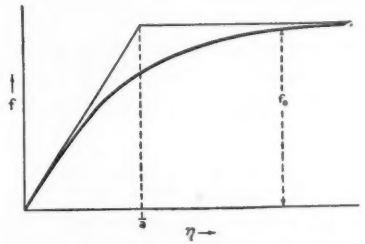

FIGURE 1

$f(\eta)$ only for smaller values of η, we may substitute (40) into (39). We shall also neglect in the first approximation the changes of ε_1 and ε_2. As a matter of fact, we shall considerably simplify our formulae, without greatly affecting their generality, by putting $\varepsilon_1 = \varepsilon_2 = 0$. Making the above substitution into (39) and substituting an equality sign for the inequality, we obtain the following equation for determination of η^* the value of η for which inequality (39) breaks down.

$$w_0 f_0 a\eta^* + 2b = 3\sqrt{b w_0 f_0 a \eta^*}. \tag{42}$$

Squaring and solving for η^* we find two positive roots

$$\eta_1^* = \frac{b}{w_0 f_0 a} \quad \text{and} \quad \eta_2^* = \frac{4b}{w_0 f_0 a}. \tag{43}$$

Since equations (43) are obtained by substituting equ. (40) into (39), and since equ. (40) holds only for $\eta < 1/a$, therefore the above argument, as well as its subsequent consequences, holds only if the larger root η_2^* is less than $1/a$, or if $4b/w_0 f_0 a < 1$. Due to the approximation used, our results hold only within a restricted range of the constants. The investigation of the more general cases will be given elsewhere. The meaning of the two roots becomes clear upon inspection of (39). It can be easily shown that for $\varepsilon_1 = \varepsilon_2 = 0$ inequal-

ity (39) holds for $\eta > \eta_2^*$ and for $\eta < \eta_1^*$, while for $\eta_1^* < \eta < \eta_2^*$ the inequality (39) is reversed. Physically, the meaning of this situation can also be easily understood. A decrease of η means a decrease of proper supervision or organization of the work of individuals of type *II*, and a corresponding decrease in output. For $\eta_1^* < \eta < \eta_2^*$, this supervision becomes too small to enable class *I* to retain a sufficient amount of goods relative to *II*. Still the work of class *II* produces enough goods to make both dW_1/dt and dW_2/dt positive. For $\eta > \eta_1^*$, both sides of (39) are positive. When η decreases still further, approaching zero, the individuals of class *II* cannot produce enough and again $dW_1/dt > dW_2/dt$, but both are negative. The first class loses less than the second.

We thus see that, since η decreases with time, following a period of "concentration" of accumulated wealth in class *I*, there is a period of "deconcentration." The beginning of the "decline" of class *I* is given by the equation

$$\eta = \eta_2^* \tag{44}$$

in which we have to substitute for η_2^* the expression (43), for η the expression (38), and then introduce for n_1' and n_2 the expressions (34) and (36). This gives:

$$\frac{\alpha n_{01}(e^{\beta t} - 1) + n_{01}}{n_{02}\, e^{\beta t}} = \frac{4b}{w_0 f_0 a}. \tag{45}$$

Solved with respect to t:

$$t = \frac{1}{\beta} \log \frac{(1 - \alpha)\, n_{01} w_0 f_0 a}{4b n_{02} - \alpha n_{01} w_0 f_0 a}. \tag{46}$$

In order that t should be real we must have

$$4b n_{02} > \alpha n_{01} w_0 f_0 a. \tag{47}$$

which imposes an upper limit on n_{01}/n_{02}. The meaning of this becomes clear when we consider that with the assumption made here about $f(\eta)$, an increase of n_{01}/n_{02} above a certain limit leaves $f(\eta)$ and therefore dW_1/dt and dW_2/dt practically unchanged. On the other hand, n_1' tends, as seen from (36) and (34), to αn_1. Hence $\eta = n_1'/n_2$ tends to the value $\alpha n_1/n_2 = \alpha n_{01}/n_{02}$. If originally there has been a very large "reserve" of n_{01}/n_{02}, then this final value of η may still be sufficient, under these conditions, to keep $dW_1/dt > dW_2/dt$. In reality such a case is very unlikely to occur. It would actually mean a very large "initial reserve" of executive and supervisory personnel. In any

case, the ratio W_1/W_2 would decrease due to the decrease of ε_1 and increase of ε_2, which we neglected in our approximation.

Moreover, we must consider that gradually the number n'_2 of individuals of type I in class II is increasing. When the ratio $\eta' = n_2'/n_2$ becomes sufficiently large, a new class of I' of "supervisors" will be formed within class II, and I' will begin to compete with class I. Depending on the choice of the constants, this may happen even before class I "declines" appreciably spontaneously.

The rise of class I' and its competition with I will begin when η' satisfies the equation

$$\begin{aligned} &\varepsilon_1\eta + w_0 f(\eta) + b - 2\sqrt{bw_0 f(\eta)} \\ &= \varepsilon_1\eta' + w_0 f(\eta') + b - 2\sqrt{bw_0 f(\eta')} , \end{aligned} \tag{48}$$

which requires that $\eta = \eta'$ and in which we have to put [because of (34) and (36)]

$$\eta = \frac{n_1'}{n_2} = \frac{\alpha n_{01}(e^{\beta t} - 1) + n_{01}}{n_{02}e^{\beta t}} ;$$

$$\eta' = \frac{n_2'}{n_2} = \frac{\alpha(e^{\beta t} - 1)}{e^{\beta t}} .$$

This gives:

$$t = \frac{1}{\beta} \log \frac{(1-\alpha)n_{01} + \alpha n_{02}}{\alpha(n_{02} - n_{01})} . \tag{50}$$

We may consider a more complicated case, in which, due to the difference of the amounts of wealth already accumulated, the competitor in order to succeed must start with more stringent conditions than (48). For instance, we may require that the right-hand side of (48) exceeds the left-hand side by an amount which is a function of the difference of accumulated W_1 and W_2. This will give for t, instead of (50), an expression which will involve such constants as ε_1, ε_2, f_0, a and b.

The interesting result of this study is that it shows us a simple picture of the "birth" of a "wealthy" class, with an ensuing initial "concentration" of "wealth," followed by a "decline" and finally succeeded by a class composed of different individuals. While a differentiation according to accumulated wealth remains, yet the class distinction shifts with time from one group of individuals to another. When developed much farther, such a study may perhaps be connected with P. Sorokin's interesting study on the "life-span" of different social organizations. Comparisons of calculated life span (e.g., equ. 46)

with the observed ones may give us the values of some of the constants involved.

The most interesting further developments, however, would be in connection with our previous study (1) of the life-span and interactions of classes, which exercise a control due to their initiative, regardless of the amount of wealth. The existence of a "wealthy" class is, after all, possible only when the majority of the individuals in class *II* regulate their behaviour according to the principles involving the sanctity of private property. Otherwise, they may take away the accumulated wealth of class *I* by sheer force, due to their numerical prevalence. Thus, the existence of such a class *I* presupposes either an influence of the individuals of this class on those of type *II*, or the existence of another "influencing" group, which determines the behaviour of individuals of type *II*. In the latter case, any changes of the interrelations between the influencing group and the rest of the population will have strong repercussions on the relation of class *I* to class *II* and on W_1/W_2. Those kinds of relations will form the object of future studies.

The reader may now better understand the reasons why we did not avail ourselves of the much more developed mathematical economic theories. We could have introduced much less schematized pictures, discussing directly questions of cost of production, overhead, marginal profits, fluctuations of demand, etc. (2, 3). But even the very schematic picture used here leads us to rather complicated equations. At this stage of the development, the introduction of the abovementioned complications would make the task hopelessly difficult. In the future, however, by successively more complex steps we should attempt to take all these very essential details into account.

II

We have considered in a previous paper various types of social interaction in which a population of individuals has been divided into two or more classes. However, hitherto we have not paid any attention to the spatial distribution of the individuals belonging to different classes. In this section we shall illustrate by a few simple examples how possible effects of different spatial distribution could be studied mathematically (6). Again we select intentionally oversimplified examples, which in their simple form may have no counterpart in reality. We are here concerned not so much with deriving equations for actual cases as with illustrating the methodological principles.

We shall discuss here a case which, while in its present form

purely theoretical and quite abstract, may in its eventual developments throw some light on the relation between rural and urban population.

Suppose that out of N individuals a number N_r draw their means of existence chiefly from land cultivation. Let them be distributed on the average uniformly through an area suitable to produce the necessary amount of land products. On the other hand, let the N_u remaining individuals gather together in cities and produce goods that do not require *directly* any land. We have

$$N = N_u + N_r . \tag{51}$$

Let the amount of goods produced by a person on the land be p_r per unit time, while the corresponding amount produced by a person in the city be p_u. In general p_r and p_u are not constant, but both may be functions of N_r and N_u, thus

$$p_u = f_u(N_u , N_r) \quad ; \quad p_r = f_r(N_u , N_r) . \tag{52}$$

Thus, for instance, for a land very scarcely populated and not used to the full extent, p_r may be approximately constant and determined by a constant fertility of the soil and by the average constant capacity of a man to produce work. However, as N_r increases, the total amount $N_r p_r$ of goods produced must necessarily tend to a constant value, determined by the *maximum possible* fertility of the soil. Hence p_r must decrease with increasing N_r. The law of decrease will of course be rather complex and depend on many factors. As an illustration, we may consider just a simple form

$$p_r = \frac{a_1}{N_r + a_2} , \tag{53}$$

where a_1 and a_2 are constants. In general, however, the value of p_r may depend also on N_u.

Now we may consider different cases of behaviour of the individuals. A simple case is when every individual tries to produce as much as he can. If for a given N_r and $N_u = N - N_r$, $p_u > p_r$, then individuals from the country will migrate to the city, until, due to the change of N_u and N_r, produced by this migration, p_u becomes equal to p_r. If $p_u < p_r$, a migration from city to the country will take place. Since, because of (51), $N_r = N - N_u$, p_u and p_r in (52) are functions of N and N_u. Hence the requirement

$$p_u = p_r \tag{54}$$

gives us an equation for the determination of N_r for a given total N.

To illustrate, let us assume that in the city p_u = Const., while

p_r is given by (53). Since the maximum value of p_r is equal to a_1/a_2, and p_r decreases monotonically with increasing N_r, in order for equation (54) to be possible at all, we must have

$$\frac{a_1}{a_2} > p_u \,, \tag{55}$$

or

$$a_2 - \frac{a_1}{p_u} < 0 \,. \tag{56}$$

Put

$$a_2 - \frac{a_1}{p_u} = -c^2 \,. \tag{57}$$

Equations (54) and (53) give:

$$N_r = \frac{a_1}{p_u} - a_2 \,, \tag{58}$$

or because of (51):

$$N_u = N + a_2 - \frac{a_1}{p_u} \,. \tag{59}$$

From (59) and (57) we have:

$$\frac{N_u}{N} = 1 - \frac{c^2}{N} \,. \tag{60}$$

Thus, the ratio of urban population to the total population increases with increasing total population N, tending to 1 for $N = \infty$.

We may consider a different mode of behavior of the individuals. For instance, as a purely theoretical example, we may consider the case where every individual tries to increase the total amount

$$G = N_u p_u + N_r p_r \tag{61}$$

of goods produced, and according to this aim chooses either the urban or the rural occupation. In this case we have to look for such a value of N_u which makes G a maximum. This is given by the equation

$$\frac{dG}{dN_u} = 0 \,, \tag{62}$$

into which we substitute for p_u and p_r the expressions (52), after expressing N_r in terms of N and N_u by means of (51).

Again using for p_r the expression (53) and putting $p_u = \text{Const.}$, equation (62) becomes:

$$\frac{d}{dN_u}\left(N_u p_u + \frac{a_1(N - N_u)}{N - N_u + a_2}\right) = 0 , \tag{63}$$

which gives, after elementary operations:

$$N_u = N + a_2 - \sqrt{\frac{a_1 a_2}{p_u}} . \tag{64}$$

From (64) we have

$$\frac{N_u}{N} = 1 + \frac{1}{N}\left(a_2 - \sqrt{\frac{a_1 a_2}{p_u}}\right) . \tag{65}$$

Because of (56) the expression in parentheses is negative. For

$$N > \sqrt{\frac{a_1 a_2}{p_u}} - a_2 . \tag{66}$$

N_u/N is positive and increases with N, becoming equal to 1 for $N = \infty$. The reversal of the inequality sign in (66) gives $N_u/N < 0$, which is physically impossible. In this case, N_u actually will be equal to zero, which means that all population in that case is rural.

We may consider a somewhat more complicated situation. Let each urban individual produce p_u and consume c_u per unit time, and each rural individual produce p_r and consume c_r per unit time. Let the rural individuals supply the urban with an amount $N_r g_r$ of rural goods and receive for each unit of those goods θ units of urban products. As in the preceding section, we have a demand function connecting g_r with θ, and we again put

$$g_r = g_0 - \frac{b}{\theta} . \tag{67}$$

Putting

$$\varepsilon_u = p_u - c_u \quad ; \quad \varepsilon_r = p_r - c_r \quad ; \quad \varepsilon = \varepsilon_r - \varepsilon_u \; ; \tag{68}$$

and using for p_r the expression (53) and again considering the case $\varepsilon_u > 0$ with $p_u = \text{Const.}$, we have:

$$\varepsilon_u = \text{Const.} > 0 \quad ; \quad \varepsilon_r = \frac{a_1}{N_r + a_2} - c_r \; ;$$

$$\varepsilon = \frac{a_1}{N_r + a_2} - c' \quad ; \quad c' = c_r + \varepsilon_u . \tag{69}$$

We have now for the rate of change of the total accumulated goods for the urban and rural population:

$$\frac{dW_u}{dt} = N_u\varepsilon_u + N_r g_r - \theta N_r g_r \ ,$$
$$\frac{dW_r}{dt} = N_r\varepsilon_r - N_r g_r + \theta N_r g_r \ . \tag{70}$$

Because of (51) and (67), (70) gives

$$\frac{dW_u}{dt} = N_u\varepsilon_u + (N - N_u)\,(g_0 - \frac{b}{\theta})\,(1 - \theta) \ ;$$
$$\frac{dW_r}{dt} = N_r\varepsilon_r - N_r(g_0 - \frac{b}{\theta})\,(1 - \theta) \ . \tag{71}$$

The per capita rates v_u and v_r are:

$$v_u = \varepsilon_u + \frac{N - N_u}{N_u}\,(g_0 - \frac{b}{\theta})\,(1 - \theta) \ ;$$
$$v_r = \varepsilon_r - (g_0 - \frac{b}{\theta})\,(1 - \theta) \ . \tag{72}$$

Consider again the case where every individual chooses his occupation in such a way as to make his *per capita* rate of accumulation as large as possible. For a fixed θ, we then have a migration either to or from the city until

$$v_u = v_r \ . \tag{73}$$

Because of (51) and (69), equation (73) gives a relation between N, N_r and θ, (or N, N_u and θ). For a fixed θ and N, it will give us the values of N_r and N_u. But, because of (72), this also fixes the values of v_u and v_r in terms of θ and N. If all individuals try to make their v_u and $v_r = v_u$ as large as possible, they will agree on such a θ as to make v_r and v_u a maximum. The details of this problem in this form present some interesting features and will be studied separately.

As emphasized above, the assumptions at the basis of these considerations are too simple and crude to lead to any results that may have applications in real cases. The general method, however, can be applied to cases of any complexity as it is hoped to be done in the future. The simple requirements (54) and (73) will have to be replaced by more complex ones, which will take into account the variability of p_u, c_u, p_r and c_r from individual to individual. This will result in the situation that, while for some individuals, for a given N_u/N, a migration to city will be advantageous, for others an oppo-

site migration will be indicated. This will complicate the problem considerably, but also will bring it closer to actual cases. A combination of the considerations of this paper with those of the previous one (1) looks particularly interesting and promising.

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